Lecture 26: Proof of Picard-Lindelöf

Theorem: Consider the IVP $y' = f(x,y)$, $y(x_0) = y_0$, $x \in \mathbb{R}, y \in \mathbb{R}^n$. Let ~~f: $\mathbb{R}\times\mathbb{R}^n$~~ $f: \mathbb{R}\times\mathbb{R}^n \to \mathbb{R}^n$ be continuous in x and uniformly Lipschitz continuous in y (i.e. $|f_x(y_1) - f_x(y_2)| \le M|y_1 - y_2|$ for some M fixed for all x). Then, there exists some $\varepsilon > 0$ such that a unique solution y exists on $[x_0 - \varepsilon, x_0 + \varepsilon]$.

Lemma: Banach Fixed Point Theorem:
Let (X,d) be a complete, non-empty metric space. Let $T: X \to X$ be such that $d(T(x), T(y)) \le q\, d(x,y)$ for $q \in [0,1)$, $x,y \in X$. Then, T admits a unique point x^* so $T(x^*) = x^*$.

$\boxed{\text{Pf}}$ Pick $y \in X$ arbitrarily. Set $y_0 = y$, and $y_{n+1} = T(y_n)$ to generate a sequence $\{y_n\}$.

Notice $d(y_n, y_k)$ for $n > k$ has

$$d(y_n, y_k) \le d(y_k, y_{k+1}) + d(y_{k+1}, y_{k+2}) \cdots + d(y_{n-1}, y_n)$$

and

$$d(y_j, y_{j+1}) = d(T^j y_0, T^{j+1} y_0) \le q^j d(y_0, y_1)$$ such that

$$d(y_n, y_k) \le d(y_0, y_1)(q^k + q^{k+1} + \dots + q^{n-1}) \le d(y_0, y_1) \cdot \frac{q^k}{1-q}$$

Since $q < 1$, the RHS $\to 0$ for large k, or $\{y_n\}$ is Cauchy. Hence, there is some limit $x^* = \lim_{n\to\infty} y_n$.

Notice $x^* = \lim_{n\to\infty} T(y_n) = T(\lim_{n\to\infty} y_n) = T(x^*)$. If y^* is another such point,

$d(y^*, x^*) = d(T(x^*), T(y^*)) \le d(x^*, y^*) \cdot q$ so $y^* = x^*$. $\square$

__Proof of Theorem:__ Consider the case $n=1$, or $y \in \mathbb{R}$.

First, we rewrite the Solution Condition as

$$y(x) = y_0 + \int_{x_0}^{x} f(s, y(s))\, ds \qquad (A)$$

So the right looks like an operator on y. We will define it to be an appropriate contraction.

Let M be the Lipschitz Constant as above. Pick $b>0$, $a = \frac{1}{2M}$. Set $C_{a,b} = [x_0-a, x_0+a] \times [y_0-b, y_0+b]$.

Set $N = \sup_{C_{a,b}} |f(x,y)|$, which is finite b/c f is jointly continuous. Reset $a = \frac{1}{2}\min\{1/M, b/N\}$. This choice allows us to define

$$T : C([x_0-a, x_0+a]; [y_0-b, y_0+b]) \to C([x_0-a, x_0+a]; [y_0-b, y_0+b])$$

~~which is well~~ by $T(\varphi)(x) = y_0 + \int_{x_0}^{x} f(s, \varphi(s))\, ds$.

This is well-defined since

$$|T(\varphi)(x) - y_0| \le \int_{x_0}^{x} |f|\, ds \le \int_{x_0}^{x} N\, ds \le N \cdot a < b.$$

Further, $|T(\varphi_1)(x) - T(\varphi_2)(x)| \le \int_{x_0}^{x} M \cdot \|\varphi_1 - \varphi_2\|_\infty\, ds \le \frac{1}{2}\|\varphi_1 - \varphi_2\|_\infty$

So that T is a contraction.

By the Banach fixed-point theorem, there is a unique y so $T(y) = y$, giving (A). $\square$